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Application of Uniform Measurement Error Distribution

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14. ABSTRACT

Although there are closed-form solutions for calculating the Probability of False Accept (PFA) and the Probability of False Reject (PFR) of normally distributed measurement errors, there lacks solutions for non-normal distributions. Extending our knowledge to measurement errors that do not follow the normal distribution is beneficial to lowering the risk of having a high PFA or PFR. This research finds that the Uniform Distribution is generally not useful for assessing PFA and PFR.

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Executive Summary

Although there are closed-form solutions for calculating the Probability of False Accept (PFA) and the Probability of False Reject (PFR) of normally distributed measurement errors, there lacks solutions for non-normal distributions. Extending our knowledge to measurement errors that do not follow the normal distribution is beneficial to lowering the risk of having a high PFA or PFR. This research finds that the Uniform Distribution is generally not useful for assessing PFA and PFR.

Table of Contents

1 Introduction

Solving the PFA and PFR for non-normally distributed measurements errors is useful because sometimes the data will not follow a normal distribution. When this is the case, different distribution curves, such as uniform or triangular, must be used to calculate the PFA and PFR.

This paper will describe and review how to theoretically calculate the PFA and PFR given a specific distribution and the associated joint probability density function (PDF). Then, assuming uniformly distributed measurement errors, we will try to find a formula for its PFA and PFR. After this is done, we will simulate and examine the calculation for uniformly distributed measurement errors using Visual Basic for Applications (VBA), in hopes of confirming our findings from our theoretical calculations of the PFA and PFR.

2 Probability of False Accept and Probability of False Reject

A False Accept is defined as the event when a unit under test (UUT) is measured to be outside calibration tolerance limits but the difference of the observed measurement results of the UUT and the Calibration Standard (CalStd or CAL) is within acceptance limits.

A False Reject is defined as the event when a UUT is measured to be within calibration acceptance limits but the difference between the observed measurement results of the UUT and the CalStd is outside tolerance limits.

Hence, the PFA is defined as the probability of making an acceptance decision when the UUT is observed and reported to be in-tolerance but is actually out-of-tolerance. The PFR is defined as probability of making a rejection decision when the UUT is observed and reported to be out-of-tolerance but is actually in-tolerance.

Here are some standard notation and variables that will be used:

e_{UUT}	=	The measurement error from the UUT.
e_{CAL}	=	The measurement error from the CalStd.
d	=	The difference of the measurement error from the UUT and the CalStd.
$-L$	=	The lower specification limit.
L	=	The upper specification limit.
$-A$	=	The lower acceptance limit.
A	=	The upper acceptance limit.

When deriving a formula for the PFA or PFR, we use elementary probability theory to describe where our d and our e_{UUT} lie. After examining the definitions of the PFA and PFR we can easily see that:

$$PFA = Prob\{(-A \le d \le A)\ AND\ [(e_{UUT} > L) OR\ (e_{UUT} < -L)]\}$$

and

$$PFR = Prob\{(-L \le e_{UUT} \le L)\ AND\ [(d > A)\ OR\ (d < -A)]\}$$

With these definitions for the PFA and PFR, only the joint PDF of d and e_{UUT} is needed to have a complete equation for the PFA and PFR. Sometimes the joint PDF will be assumed or have to be derived.

The joint PDF is defined as:

$$f(d, e_{UUT}) = f(d|e_{UUT}) \cdot f(e_{UUT}),$$

where

$f(d|e_{UUT})$ = The conditional PDF of d given e_{UUT}

$f(e_{UUT})$ = The PDF of e_{UUT}.

With this information the PFA and PFR become:

$$PFA = \int_{L}^{\infty}\int_{-A}^{A} f(d, e_{UUT})\ dd de_{UUT} + \int_{-\infty}^{-L}\int_{-A}^{A} f(d, e_{UUT})\ dd de_{UUT}$$

and

$$PFR = \int_{-L}^{L}\int_{A}^{\infty} f(d, e_{UUT})\ dd de_{UUT} + \int_{-L}^{L}\int_{-\infty}^{-A} f(d, e_{UUT})\ dd de_{UUT}$$

4 Uniform Distribution

The Uniform Distribution has a constant PDF that forms a rectangle. The PDF is given by:

$$f(x) = \begin{cases} \frac{1}{2a}, & -a \le x \le a \\ 0, & otherwise \end{cases}$$

with variance and uncertainty as follows:

$$variance(x) = \frac{a^2}{3}$$

$$uncertainty(x) = \sqrt{variance(x)} = \frac{a}{\sqrt{3}}$$

The PDF for the Uniform Distribution can also be:

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & otherwise \end{cases}$$

With variance and uncertainty as follows:

$$variance(x) = \frac{(b-a)^2}{12}$$

$$uncertainty(x) = \sqrt{variance(x)} = \frac{b-a}{\sqrt{12}}$$

5 Monte Carlo Simulation of Uniform Distribution

Through VBA, a simulation can be developed that calculates both the PFA and the PFR of measurements errors with a Uniform Distribution. We will define the two functions that calculate the PFA and PFR as $USimFa$ and $USimFr$, respectively. Each of these functions will take the same 5 inputs:

b	=	The UUT limit.
c	=	The CAL limit.
A	=	The tolerance limit.
L	=	The acceptance limit.
$simCnt$	=	The number of times the simulation will repeat itself.

Within each function, there are several variables, which will be used in each iteration to calculate the number of times that the difference of the measurement error of the UUT and CalStd (d) and the measurement error of the UUT (e_{UUT}) lie inside or outside the correct intervals necessary to be a false accept or false reject. To insure each simulation is random, there will be a Randomize (Rnd) function that randomly chooses a number between 0 and 1, which is used to calculate the e_{UUT}, e_{CAL}, and d. Each time the correct requirements are satisfied for PFA and PFR, the $SumFa$ or $SumFr$ will be increased by 1. Then the loop will start again with a new random number. Once the $simCnt$ is reached, the PFA or PFR will be calculated by dividing $SumFa$ or $SumFr$ by $simCnt$ to obtain PFA or PFR, respectively. The VBA code for this simulation is as follows:

```
Option Explicit

Function USimFa(b As Double, c As Double, A As Double, L As Double, simCnt As Long) As Double
    'b is UUT limit
    'c is CAL limit
    'A is acceptance limit
    'L is specification limit

    Dim eUUT As Double
    Dim eCAL As Double
    Dim d As Double
    Dim FaAvg As Double
    Dim SumFa As Double
    Dim Fa As Long
    Dim i As Long
    SumFa = 0

    For i = 1 To simCnt

        'initialize random number generation
        Randomize

        eUUT = (Rnd() * 2 * b) - b
        eCAL = (Rnd() * 2 * c) - c
        d = eUUT - eCAL

        If (d >= -A And d <= A) And (eUUT > L Or eUUT < -L) Then
            Fa = 1
        Else
            Fa = 0
        End If

        SumFa = SumFa + Fa
    Next

    USimFa = SumFa / simCnt

End Function
```

and

```
Option Explicit

Function USimFr(b As Double, c As Double, A As Double, L As Double, simCnt As Long) As Double
```

```
    'b is UUT limit
    'c is CAL limit
    'A is acceptance limit
    'L is specification limit

    Dim eUUT As Double
    Dim eCAL As Double
    Dim d As Double
    Dim FaAvg As Double
    Dim SumFr As Double
    Dim Fr As Long
    Dim i As Long
    SumFr = 0

    For i = 1 To simCnt

        'initialize random number generation
        Randomize

        eUUT = (Rnd() * 2 * b) - b
        eCAL = (Rnd() * 2 * c) - c
        d = eUUT - eCAL

        If (eUUT >= -L And eUUT <= L) And (d > A Or d < -A) Then
            Fr = 1
        Else
            Fr = 0
        End If

        SumFr = SumFr + Fr
    Next

    USimFr = SumFr / simCnt

End Function.
```

With these two functions we can also create a Subroutine that is assigned to a button, which calculates the PFA and PFR from given inputs to the Excel worksheet. The Excel worksheet would look as follows:

	A	B	C	D	E	F	G	H	I
1				Lower	Upper				
2		UUT Limits					Calculate PFA and PFR		
3		CAL Limits							
4		Tolerance Limits							
5		Acceptance Limits							
6		Simulation Count							
7									
8		**PFA**							
9		**PFR**							

The VBA code assigned to the button would be the following:

```
Option Explicit

Sub Calculate()

    Dim dUUTLimLow As Double
    Dim dUUTLimUp As Double
    Dim eUUTLimLow As Double
    Dim eUUTLimUp As Double
    Dim TolLow As Double
    Dim TolUp As Double
    Dim AcceptLow As Double
    Dim AcceptUp As Double
    Dim simCnt As Long

    Dim rngInput As Range
    Set rngInput = Range("D2")

    dUUTLimLow = rngInput(1, 1)
    dUUTLimUp = rngInput(1, 2)
    eUUTLimLow = rngInput(2, 1)
    eUUTLimUp = rngInput(2, 2)
    TolLow = rngInput(3, 1)
    TolUp = rngInput(3, 2)
    AcceptLow = rngInput(4, 1)
    AcceptUp = rngInput(4, 2)
    simCnt = rngInput(5, 1)

    Dim rngOutput As Range
    Set rngOutput = Range("D8")

    Dim dPFA As Double
    dPFA = USimFa(dUUTLimUp, eUUTLimUp, TolUp, AcceptUp, simCnt)
    rngOutput(1, 1) = dPFA
```

```
    Dim dPFR As Double
    dPFR = USimFr(dUUTLimUp, eUUTLimUp, TolUp, AcceptUp, simCnt)
    rngOutput(2, 1) = dPFR

End Sub.
```

6 Simulation Results

Here are some results of calculating the PFA and PFR with different limits.

	A	B	C	D	E	F	G	H	I
1				Lower	Upper				
2		UUT Limits		-10	10				
3		CAL Limits		-5	5			Calculate PFA and PFR	
4		Tolerance Limits		-4	4				
5		Acceptance Limits		-6	6				
6		Simulation Count		1000000					
7									
8		**PFA**		0.044301					
9		**PFR**		0.253253					

	A	B	C	D	E	F	G	H	I
1				Lower	Upper				
2		UUT Limits		-1	1				
3		CAL Limits		-0.25	0.25			Calculate PFA and PFR	
4		Tolerance Limits		-1	1				
5		Acceptance Limits		-1	1				
6		Simulation Count		1000000					
7									
8		**PFA**		0.000000					
9		**PFR**		0.065935					

The second simulation had the surprising result that the PFA was exactly 0%. With the Uniform Distribution, 100% of the UUT measurements errors should occur between the limits of the UUT. If these limits are the same as or less than the tolerance limits, no measurement errors will ever be out-of-tolerance. As a result, there will never be a false accept. Hence, the PFA would be 0%, which the simulation confirms

Also, in this situation, no UUT measurement errors will ever be out-of-tolerance. Therefore, all observed out-of-tolerances will be false rejects caused by the error in the calibrator. This means that the PFR will not be 0%, which the simulation confirms.

It should be noted here, though, that this result calls into question whether the Uniform Distribution is useful for assessing calibration decision risk.

7 Conclusions

Although finding a closed-form solution for a measurement error that follows the Uniform Distribution amounts to finding the correct joint PDF is difficult, writing a simulation with VBA that simulates the Uniform Distribution was quite easy. An attempt to find the joint PDF of two uniform random variables was not successful. However, the simulation gives accurate and valuable information for assessing the risk associated with measurement errors following the Uniform Distribution.

Generally, risk assessment is used in situations where genuine out of tolerance results are possible. Though the Uniform Distribution is a good model for some kinds of measurement error (e.g., resolution), it would be rare that a calibration measurement error would be bounded completely by known limits as required by this distribution. So, while the Uniform Distribution can be used to get conservative measurement uncertainty estimates, it is suggested that this distribution is generally not appropriate for assessing measurement risk.

References

1. *Handbook for the Application of ANSI/NCSL Z540.3-2006 – Requirements for the Calibration of Measuring and Test Equipment*, 2009 NCSLI International.